

The challenge of non-hermitian structures in physics

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We present a brief review of physical problems leading to indefinite Hilbert spaces and non-hermitian Hamiltonians. With the exception of pseudo-Riemannian manifolds in GR, the problem of a consistent physical interpretation of these structures still waits to be faced.

I. INTRODUCTION

Among attempts to generalize the orthodox quantum theory, a visible place belongs to the indefinite metric and non-hermitian Hamiltonians. The indefinite Hilbert spaces were proposed in 1942 by Dirac [1]. In 1950 Gupta applies the idea in QED to avoid the negative energy photons [2]. The pseudo-euclidean structures are generic in relativistic theories. The mathematical studies of Pontriagin [3], Kreĭn et al. [4], [5], [6], have offered the canonical forms of the pseudo-hermitian operators (see also [7], [8]). A notable physical case is the classification of the real, 4×4 pseudo-hermitian energy momentum tensor T^α_β in terms of the “null-legs” [9].

The non-hermitian Hamiltonians (in genuine Hilbert spaces) appear in problems of the continuous measurements [10], [11], [12] and “time operator” [13], in some works on superconductors and other applied studies [14], [15], [16]. As to the indefinite Hilbert spaces, they awake a comprehensible distrust due to the “ghosts” of negative probabilities (in some occasions, though, enjoying better press than the negative energies [17], [18]!). On the other hand, Bender et al. [19]–[23] consider the non-hermitian Hamiltonians with real spectra, enjoying the \mathcal{PT} symmetry: $H^\dagger = \mathcal{PT}H\mathcal{PT}$; the aspect of indefinite metric being studied by Znojil [24]. In all these attempts the challenge of consistent physical interpretation persists [25]. To illustrate it, we quote briefly the main structural facts concerning the pseudo-hermitian operators in indefinite Hilbert spaces.

II. PSEUDO-HERMITIAN OPERATORS

Let X be a complex linear space, $\dim X = N < +\infty$. The mapping $X \ni x, y \rightarrow \langle x, y \rangle \in \mathbb{C}$ is called an *indefinite scalar product* if:

- i.* $\langle x, \alpha_1 y_1 + \alpha_2 y_2 \rangle = \alpha_1 \langle x, y_1 \rangle + \alpha_2 \langle x, y_2 \rangle$
- ii.* $\langle x, y \rangle = \langle y, x \rangle^*$
- iii.* $\langle x, y \rangle = 0 \quad \forall x \quad \Rightarrow \quad y = 0$
- iv.* $\exists x \neq 0$ such that $\langle x, x \rangle = 0$

(1)

The space X with an indefinite scalar product is called *pseudo-euclidean*, or *indefinite Hilbert space*.

A vector $x \in X$ such that $\langle x, x \rangle = 0$ is called a *null vector*. A vector $x \in X$ is called *positive* (*negative*) if $\langle x, x \rangle > 0$ (< 0).

Two vectors $x_1, x_2 \in X$ are called *orthogonal* if $\langle x_1, x_2 \rangle = 0$. Every null vector is orthogonal to itself. Two subspaces $Z_1, Z_2 \subset X$ are *orthogonal* if $\langle z_1, z_2 \rangle = 0$ for every $z_1 \in Z_1$ and $z_2 \in Z_2$. A subspace $V \subset X$ such that $\langle u, v \rangle = 0$ for all $u, v \in V$ is called a *null subspace*.

The subspace $Y \subset X$ is called *positive* (*negative*) if (i) $\langle x, x \rangle \geq 0$ (≤ 0) for every $x \in Y$, (ii) $x \in Y$ and $\langle x, x \rangle = 0$ implies $x = 0$. In all orthogonal decompositions $X = X_+ \oplus X_-$ into the positive and negative subspaces X_+ and X_- , the numbers $k = \dim X_+$ and $l = \dim X_-$ are the same. The pair (k, l) , $k + l = N$ corresponds to the numbers of positive and negative vectors in any orthogonal basis and is called the *signature* of X . Notice that the subspace $V \subset X$ is a null subspace iff V is orthogonal to itself. The dimensions of null subspaces in X are limited by the following

Lemma 1 (Pontriagin) The maximal possible number of linearly independent, mutually orthogonal null vectors in an indefinite Hilbert space of signature (k, l) is $r = \min(k, l)$.

A subspace $Y \subset X$ is called *nonsingular* if $Y \cap Y^\perp = \{0\}$ and *singular* if the intersection $Y \cap Y^\perp$ contains at least one non-zero vector. In particular, every positive (negative) subspace is nonsingular. The whole X is nonsingular due to (1)iii.

For every linear operator A defined in X there exists exactly one operator A^\dagger such that

$$\langle x, Ay \rangle = \langle A^\dagger x, y \rangle \quad (2)$$

The operator A is called *hermitian with respect to* \langle, \rangle (or *pseudo-hermitian*) if $A^\dagger = A$. For an indefinite product \langle, \rangle , several important properties of the traditional self-adjoint operators do not hold: (a) the pseudo-hermitian operators can have complex eigenvalues; (b) they do not need to be diagonalizable; (c) the eigenvectors are not necessarily orthogonal. All this can happen due to the existence of non-trivial null vectors and nilpotent operators in X . The easiest examples illustrating (a)–(c), are constructed with the help of the *dyadic operators* $u \otimes v = |u\rangle\langle v|$ defined by

$$(u \otimes v)x = \langle v, x \rangle u \quad (x \in X) \quad (3)$$

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Obviously, $(u \otimes v)^\dagger = v \otimes u$. Choosing now two null vectors $v, \tilde{v} \in X$ with $\langle v, \tilde{v} \rangle = 1$ and putting

$$A = v \otimes \tilde{v} \quad \text{and} \quad B = i(\tilde{v} \otimes v - v \otimes \tilde{v}) \quad (4)$$

one has $A^\dagger = A$, $B^\dagger = B$. Yet, $A^2 = 0$ though $A \neq 0$; hence, A is non-diagonalizable. Moreover, $Bv = -iv$ and $B\tilde{v} = i\tilde{v}$; hence B possesses the imaginary eigenvalues for two non orthogonal eigenvectors v and \tilde{v} . As one can show, A and B are the simplest bricks of the non-trivial pseudo-hermitian structures. In fact, the Jordan's cell of any real eigenvalue λ of the hermitian $A = A^\dagger$ hosts a nilpotent hermitian operator $Q = A - \lambda$, $Q^q = 0$, $Q^{q-1} \neq 0$. If $2s \leq q$, the last s vectors in the Jordan's chain $x, Qx, \dots, Q^{q-1}x$ must be null and mutually orthogonal (indeed, $i + j \geq q \Rightarrow \langle Q^i x, Q^j x \rangle = \langle x, Q^{i+j} x \rangle = 0$). The indefinite metric permits the existence of such chains, though the Pontriagin's lemma restricts their length to $s \leq r$. By choosing adequately the initial vector x one reduces the entire chain either to a "null leg" $v_1, \dots, v_s, \tilde{v}_1, \dots, \tilde{v}_s$ (where $\langle v_i, v_j \rangle = \langle \tilde{v}_i, \tilde{v}_j \rangle = 0$, $\langle v_i, \tilde{v}_j \rangle = \pm \delta_{ij}$) or to a null leg plus a unit vector e ($\langle e, v_i \rangle = \langle e, \tilde{v}_i \rangle = 0$, $|\langle e, e \rangle| = 1$) $v_1, \dots, v_s, e, \tilde{v}_s, \dots, \tilde{v}_1$; a basis which brings Q to one of the operational schemes:

$$v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_s \rightarrow \tilde{v}_s \rightarrow \dots \rightarrow \tilde{v}_1 \rightarrow 0$$

$$v_1 \rightarrow \dots \rightarrow v_s \rightarrow e \rightarrow \tilde{v}_s \rightarrow \dots \rightarrow \tilde{v}_1 \rightarrow 0$$

with obvious canonical forms:

$$\begin{aligned} \pm Q = & v_2 \otimes \tilde{v}_1 + v_3 \otimes \tilde{v}_2 + \dots + v_s \otimes \tilde{v}_{s-1} \\ & \tilde{v}_s \otimes \tilde{v}_s + \tilde{v}_1 \otimes v_2 + \dots + \tilde{v}_{s-1} \otimes v_s \end{aligned} \quad (5)$$

$$\begin{aligned} \pm Q = & v_2 \otimes \tilde{v}_1 + \dots + v_s \otimes \tilde{v}_{s-1} + e \otimes \tilde{v}_s \\ & + \tilde{v}_s \otimes e + \tilde{v}_1 \otimes v_2 + \dots + \tilde{v}_{s-1} \otimes v_s \end{aligned} \quad (6)$$

The null legs are crucial as well for the complex eigenvalues. Indeed, if $A^\dagger = A$, each complex eigenvalue $\lambda = \alpha + i\beta$ is accompanied by λ^* of equal multiplicity. Instead of analyzing separately the Jordan's subspaces of λ and λ^* it is more convenient to determine the structure of A in their sum $X_{(\lambda \lambda^*)} = X_{(\lambda)} + X_{(\lambda^*)}$ by using the nilpotent pair $Q = A - \lambda$, $Q^\dagger = A - \lambda^*$. Since Q and Q^\dagger are both nilpotent in $X_{(\lambda \lambda^*)}$, so is QQ^\dagger . As one can show, there must exist a vector $x \in X_{(\lambda \lambda^*)}$ such that the triangle Δ of vectors

$$\begin{array}{ccccc} & & x & & \\ & Qx & & Q^\dagger x & \\ Q^2 x & & QQ^\dagger x & & Q^{\dagger 2} x \\ & & \dots & & \\ & Q^{2s-1} x & \dots & Q^s Q^{\dagger s-1} x & Q^{s-1} Q^{\dagger s} x \dots Q^{\dagger 2s-1} x \end{array} \quad (7)$$

spans a nonsingular subspace $X_\Delta \subset X_{(\lambda \lambda^*)}$, where Q^{2s} and $Q^{\dagger 2s}$ vanish but not Q^{2s-1} and $Q^{\dagger 2s-1}$. The $2s$ vectors of the last row are linearly independent and form

a natural basis in the triangle subspace X_Δ . Now, it is easy to show that by choosing properly the top vector $x \in X_\Delta$ one can reduce the basic row of (7) to a null leg defined as

$$\begin{array}{cccc} Q^{2s-1}x, & \dots, & Q^s Q^{\dagger s-1}x, & Q^{s-1} Q^{\dagger s}x, & \dots, & Q^{\dagger 2s-1}x \\ \parallel & & \parallel & & \parallel & \parallel \\ \tilde{v}_1 & & \tilde{v}_s & & v_1 & v_s \end{array} \quad (8)$$

Observing the action of Q and Q^\dagger on (8) and remembering that $Q - Q^\dagger = 2i\beta$, $Q^{2s} = Q^{\dagger 2s} = 0$ in X_Δ , one sees that A in X_Δ has the canonical form:

$$\begin{aligned} A_\Delta = & \alpha \sum_{j=1}^s (\tilde{v}_j \otimes v_j + v_j \otimes \tilde{v}_j) + \\ & i\beta \sum_{j=1}^s (\tilde{v}_j \otimes v_j - v_j \otimes \tilde{v}_j) + \\ & 2i\beta \sum_{j=1}^{s-1} \sum_{i=j+1}^s (\tilde{v}_j \otimes v_i - v_i \otimes \tilde{v}_j) \end{aligned} \quad (9)$$

where $2s = \dim X_\Delta \leq 2r$ due to lemma 1. As one can easily show, each $X_{(\lambda \lambda^*)}$ decomposes into an orthogonal sum of nonsingular triangular subspaces X_Δ where A acquires the form (9), so we have

Theorem The hermitian operator A in a pseudo-euclidean space X is reducible to the sequence of Jordan's cells corresponding to real eigenvalues λ_i where $Q_i = A - \lambda_i$ are of the canonical form (5-6), and to a number of cells of the complex $\lambda_i \lambda_i^*$ with A given by (9), the total number of mutually orthogonal null vectors in all irreducible cells of type (5), (6), (9) being limited by the Pontriagin criterion.

III. THE PHYSICAL PROBLEMS

The pseudo-hermitian structures have a well defined status in classical theories. Thus, e.g., the energy momentum tensor T_β^α of GR is an example of a hermitian operator in the pseudo-euclidean space of signature $(+ - - -)$. So, according to the canonical forms of Sec. II, it can adopt only 4 basic types of Plebanski [9]: (1) $[Z - Z^* - S_1 - S_2]$ (diagonalizable, real eigenvalues S_1, S_2 , complex ones Z, Z^*); (2) $[T - S_1 - S_2 - S_3]$ (complete diagonalization with four linearly independent eigenvectors); (3) $[2N - S_1 - S_2]$ (non-diagonalizable canonical form which admits three eigenvectors); (4) $[3N - S_1]$ (non-diagonalizable canonical form admitting only two eigenvectors). The analogous types would exist in more dimensions for the signature $(1, l)$.

An interesting case of *strings* living in a pseudo-euclidean space \mathbb{R}^4 of signature $(+ + - -)$ is discussed in [26]. Notice that the field theories formulated in such space would lead to the energy momentum tensors of new algebraic types. Generalizing [9], all of them can be reduced to the following standard forms:

(1) $[2Z - 2Z^*]$; two complex eigenvalues with nontrivial Jordan's cells; the canonical form $A = \alpha[(\tilde{v}_1 \otimes v_1 + v_1 \otimes$

$\tilde{v}_1) + (\tilde{v}_2 \otimes v_2 + v_2 \otimes \tilde{v}_2)] + i\beta[(\tilde{v}_1 \otimes v_1 - v_1 \otimes \tilde{v}_1) + (\tilde{v}_2 \otimes v_2 - v_2 \otimes \tilde{v}_2)] + 2i\beta(\tilde{v}_1 \otimes v_2 - v_2 \otimes \tilde{v}_1);$

(2) $[Z_1 - Z_1^* - Z_2 - Z_2^*]$, two pairs of complex conjugate eigenvalues, complete diagonalization: $A = \alpha_1(\tilde{v}_1 \otimes v_1 + v_1 \otimes \tilde{v}_1) + i\beta_1(\tilde{v}_1 \otimes v_1 - v_1 \otimes \tilde{v}_1) + \alpha_2(\tilde{v}_2 \otimes v_2 + v_2 \otimes \tilde{v}_2) + i\beta_2(\tilde{v}_2 \otimes v_2 - v_2 \otimes \tilde{v}_2);$

(3) $[Z_1 - Z_1^* - 2S_1]$ with $A = \alpha_1(\tilde{v}_1 \otimes v_1 + v_1 \otimes \tilde{v}_1) + i\beta_1(\tilde{v}_1 \otimes v_1 - v_1 \otimes \tilde{v}_1) \pm (\tilde{v}_2 \otimes \tilde{v}_2) \pm \lambda_2(v_2 \otimes \tilde{v}_2 + \tilde{v}_2 \otimes v_2);$

(4) $[Z_1 - Z_1^* - S_1 - S_2]$ with $A = \alpha_1(\tilde{v}_1 \otimes v_1 + v_1 \otimes \tilde{v}_1) + i\beta_1(\tilde{v}_1 \otimes v_1 - v_1 \otimes \tilde{v}_1) \pm \lambda_2(e_2 \otimes e_2) \pm \lambda_3(e_3 \otimes e_3);$

(5) $[T_1 - T_2 - S_1 - S_2]$ with the trivial $A = \lambda_1(e_1 \otimes e_1) + \lambda_2(e_2 \otimes e_2) - \lambda_3(e_3 \otimes e_3) - \lambda_4(e_4 \otimes e_4)$

(6) $[2N_1 - 2N_2]$ with $A = \pm(\tilde{v}_1 \otimes \tilde{v}_1) \pm \lambda_1(v_1 \otimes \tilde{v}_1 + \tilde{v}_1 \otimes v_1) \pm (\tilde{v}_2 \otimes \tilde{v}_2) \pm \lambda_2(v_2 \otimes \tilde{v}_2 + \tilde{v}_2 \otimes v_2);$

(7) $[2N - S_1 - S_2]$ with $A = \pm(\tilde{v}_1 \otimes \tilde{v}_1) \pm \lambda_1(v_1 \otimes \tilde{v}_1 + \tilde{v}_1 \otimes v_1) + \lambda_2(e_2 \otimes e_2) + \lambda_3(e_3 \otimes e_3);$

(8) $[3N - S_1]$ with $A = \pm(e \otimes \tilde{v}_1 + \tilde{v}_1 \otimes e) \mp \lambda_1(v_1 \otimes \tilde{v}_1 + \tilde{v}_1 \otimes v_1 + e \otimes e) \mp \lambda_2(e_2 \otimes e_2);$

(9) finally $[4N]$ with $A = \pm(v_2 \otimes \tilde{v}_1 + \tilde{v}_2 \otimes \tilde{v}_2 + \tilde{v}_1 \otimes v_2) \pm \lambda(v_1 \otimes \tilde{v}_1 + \tilde{v}_1 \otimes v_1 + v_2 \otimes \tilde{v}_2 + \tilde{v}_2 \otimes v_2).$

(where $|\langle e_i, e_j \rangle| = |\langle v_i, \tilde{v}_j \rangle| = \delta_{ij}$, $\langle v_i, e \rangle = \langle \tilde{v}_i, e \rangle = \langle v_i, v_j \rangle = \langle \tilde{v}_i, \tilde{v}_j \rangle = 0$). The question of physical nature of the corresponding sources remains open.

In quantum theories the main challenge is caused by negative vectors (“ghosts”), as well as by the absence of a consistent measurement theory. Thus, e.g. in an interesting plasma study ([15]) it is assumed that the eigenvectors of the pseudo-hermitian operator must form a basis in X . However, it is not so: As we have seen, even in the signature $(1, l)$ the subspaces of real eigenvalues may have a nontrivial Jordan structure.

In an ample class of quantum field theories following

Gupta and Dirac [1], [2] the indefinite Hilbert spaces (with $\dim X = +\infty$) arise as an auxiliary element, eliminated later by the constraints conditions to avoid the “ghost” vectors. The original field operators turn as well “ghost observables” to be substituted by the constrained ones [27], [28], [29]. However, once all ghosts depart, an unsolved mystery remains why the ghost formulation of the theory was at all necessary?

The situation is different in several areas of QM showing a ‘non-hermitian dissidence’ which cannot be maintained on purely ghost level. Notice the fundamental role of non-hermitian Hamiltonians (in the orthodox Hilbert spaces) for the continuous reduction processes [10], [11], [12], [13]. For different reasons the complex Hamiltonians with real spectra are studied in [19]–[23]. Here, the story develops on the heuristic level of \mathcal{PT} -symmetric operators. In mathematical terms, it can be viewed as a new branch of the spectral analysis in Banach spaces. However the link with the pseudo-euclidean structures was recently reported by Znojil [24]. Independent steps in the same direction are taken by Takook [30]. In all these designs the consistent statistical interpretation is still missing. So, will the indefinite Hilbert spaces contribute only to a new “ghost story” or are they a real escape route from too much orthodoxy in quantum theory?

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